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COVID-19 Calculations Using the SIR-Model

Preliminary remarks

At the beginning of an epidemic typically an exponential spread of the disease or infection occurs, which means that hospitals can quickly become overloaded . The growth essentially depends on the transmission rate and the duration of an infection.

To reduce the spreading speed of Corona, consequently in almost all countries of the world more or less radical measures like mask duty, contact restrictions, curfews or even total lock-downs were applied. To find the most acceptable ways and the best decisions during an epidemic or even pandemic, mathematical models can help to better understand the exponential growth under different conditions and disposed measures. The Robert Koch Institute also uses computer models for influenza simulations and now also for COVID-19 . This can answer questions like:

- What is the maximum number of people, who can be infected at the same time and when will this be?
- What is the proportion of the population that will be infected in the worst case?
- What can be useful measures to reduce the spread?
- What is the influence of vaccination and what vaccination readiness is required?
- How many deaths have to be expected depending on the measures?

First mathematical models go back to the work of Ogilvy Kermack and Anderson Gray McKandrick in the 1927, who had developed the *Susceptible-Infected-Removed Model* (SIR-model), which got its name from the grouping of the population in **Susceptible individuals** (S), **Infected individuals** (I) and **Resistant individuals** (R) after an infection. Despite this comparatively simple model they could well model the data of a plague epidemic in Bombay in 1905-06.

The following considerations are based on the SIR model. But the basic model has been expanded by some additional functions and calculations to simulate the COVID-19 pandemic, This includes:

- Consideration of the death rate and display of the total number of deceased at the end of the calculation period,
- Impact of vaccinations under the influence of the population's willingness to be vaccinated,
- Lockdown simulation with a reduced infection rate,
- Simulation of virus mutations with changed infection or incubation time.

Basics of the model

In the SIR model the population is divided into three groups:

- **Susceptible individuals S (t),** who are at time t healthy but not immune,
- **Infected individuals I (t)** , who are contagious over the duration of infection and can transfer the virus,
- **Resistant individuals R (t)** who have become immune after the disease.

It is assumed that:

- each individual can only be infected once by a pathogen and afterwards is immune or dies, that
- infected are immediately contagious and that
- the infection-, healing-, vaccination- and death rates are assumed to be constant over the considered period.

In order to better understand the following balance equations, which are intended to mathematically map the infection process, we will start with the simplest case.

Purely exponential growth:

In average an infected person may infect 2 new persons within 1 week. The newly infected, for their part, further infect other previously uninfected people with the same probability. If all remain infectious, the number of infected people increases, starting from a single infection at time t = 0, i.e. I (0) = 1, over the time interval Δt of one week in the form:

Start: I (0) = 1,

1st week: I (1W) = I (0) + 2 = 3,

2nd week: I (2W) = I (1W) + 2*I (1W) = (1+2)*I (1W) = 3*3 = 9,

3rd week: I (3W) = I (2W) + 2*I (2W) = 3*I (2W) = 3*9 = 27,

...

20th week: I (20W) = I (19W) + 2*I (19W) = (1+2)*I (19W) = 3*1.162 billion = 3.487 billion totally infected.

This is the formation of an exponential increase. After 20 weeks, almost half the world's population is infected.

We generally designate the infection or contagion rate with α. In the case of two further new infections per week from any infected person, the infection rate is α = 2/week = 0.286/day. In the observed time interval Δt = 1 week, then the balance at time t_{n+1} is determined by the balance at time t_n :

$$I(t_{n+1}) = I(t_n + \Delta t) = I(t_n) + \alpha \cdot I(t_n) \cdot \Delta t \tag{1}$$

This equation already specifies the numerical calculation mode that is suitable to gradually calculate the number of infections at a later time t_{n+1} from a previous value at time t_n. The changing infections over the period Δt we designate as $\Delta I(t_n) = I(t_{n+1}) - I(t_n)$. Then according to Eq. (1) it holds:

$$\frac{\Delta I(t_n)}{\Delta t} = \alpha \cdot I(t_n) . \tag{2}$$

For infinitesimally small time steps with $\Delta t \rightarrow dt$ and $\Delta I \rightarrow dI$ this becomes a simple differential equation of the form:

$$\frac{dI(t)}{dt} = \alpha \cdot I(t) , \tag{3}$$

whose solution is an exponential function to the basis of the natural number e = 2.71:

$$I(t) = I(0) \cdot e^{\alpha \cdot t} . \tag{4}$$

Growth with limited duration of infection

Fortunately, after a finite period, which is normally determined by the disease and the course of the disease, an infected person is no longer contagious and can be classified as resistant or deceased. This acquired immunity significantly weakens the epidemic in two ways. A first aspect is that, for example, if the infection lasts 1 week, during this time an infected person infects 2 new people, but after his recovery is no longer contagious himself. With the initial condition I (0) = 1, then a modification of the above example applies:

I (0) = 1

1st week: I (1W) = I (0) + (2-1)*I (0) = 2,

2nd week: I (2W) = I (1W) + (2-1)*I (1W) = 4,

3rd week: I (3W) = I (2W) + (2-1)*I (2W) = 8,

...

20th week: I (20W) = I (19W) + (2-1)*I (19W) = 1,049 million.

Hence, also from this still follows an exponential increase, but not as dramatically as in the previous case without limited infection. The reciprocal value of the duration of the infection is known as the cure rate or recovery rate and is abbreviated as β. In this example, β = 1/week = 0.143/day. Thus, in analogy to Eq. (2), the increase or decrease in infections over the interval Δ t can generally be written as:

$$\frac{\Delta I(t_n)}{\Delta t} = \alpha \cdot I(t_n) - \beta \cdot I(t_n) . \tag{5}$$

So there exists an increase rate, described by $\alpha \cdot I(t)$, and a decay rate $\beta \cdot I(t)$, both proportional to the current number of infections. When $\alpha = \beta$, the balance is zero and there is no change over time.

The ratio of α to β is referred to as the base reproduction rate or R_0-value with

$$R_0 = \frac{\alpha}{\beta}. \tag{6}$$

This value is often used as a key figure for the spread of an epidemic. For $R_0 = 1$, there is no further spread of an epidemic.

A second essential aspect for the self-limiting spread of infections is that after persons recovered, generally they are not infected again, which means that the number of susceptible persons S(t) is reduced. The number of resistant individuals R(t) increases in the same way. For the balance of the infected individuals this can be expressed as a modified infection rate α', which decreases with the ratio of susceptible persons S(t) to the total population N(t) as $\alpha' = \alpha \cdot S(t)/N(t)$. This changes the increase or decrease of infected people and modifies Eq. (5):

$$\frac{\Delta I(t_n)}{\Delta t} = \alpha \cdot \frac{S(t_n)}{N(t_n)} \cdot I(t_n) - \beta \cdot I(t_n). \tag{7}$$

To the same extent, how infections are increasing per time increment and are described by the 1st term on the right-hand side of Eq. (7), the number of susceptible individuals decreases accordingly. So, $S(t_n)$ is not a constant, and $N(t_n)$ also changes over time due to the deceased. But the decrease of endangered ones per time interval must be the same as the increase in infected people over this period. Thus, the second important balance for $S(t_n)$ is:

$$\frac{\Delta S(t_n)}{\Delta t} = -\alpha \cdot \frac{I(t_n)}{N(t_n)} \cdot S(t_n). \tag{8}$$

Finally, it must be taken into account that the number of resistant persons $R(t_n)$ increases with the number of infected people who have successfully survived an infection. That is, R(t) increases proportionally with $\beta \cdot I(t)$, but has to be corrected for the number of deceased. With a death rate γ for corona infection, the balance for resistant people is:

$$\frac{\Delta R(t_n)}{\Delta t} = \beta \cdot I(t_n) - \gamma \cdot I(t_n). \tag{9}$$

Furthermore, the boundary condition for the total population $N(t_n)$ is:

$$N(t_n) = S(t_n) + I(t_n) + R(t_n). \tag{10}$$

The balance equations (7), (8) and (9), together with the boundary condition (10), form a linearly coupled system of rate equations that can be solved for the three groups in time steps of Δt. For differential time steps dt we speak of a linearly coupled differential equation system.

Influence of vaccinations

As an extension of the SIR-model we consider the influence and effect of vaccinations. For this it is assumed that a vaccinated person can no longer be infected. With a vaccination rate δ, which relates to the total population, and with an efficiency ε of an individual vaccination this results in a decrease of susceptible persons

$$\frac{\Delta S(t_n)}{\Delta t} = -\alpha \cdot \frac{I(t_n)}{N(t_n)} \cdot S(t_n) - \delta \cdot \varepsilon \cdot N(t_n). \tag{11}$$

In the same way, as the number of susceptible people decreases, is the number of resistant ones increasing, and equation (9) must be replaced by

$$\frac{\Delta R(t_n)}{\Delta t} = (\beta - \gamma) \cdot I(t_n) + \delta \cdot \varepsilon \cdot N(t_n). \tag{12}$$

So, a coupled system of equations is available for calculating the COVID-19 pandemic, which can be solved with sufficient time resolution in steps of $\Delta t = 0.1$ days using an Excel spreadsheet program for the three categories:

- **Susceptible Individuals S(t),**
- **Infected Individuals I(t),**
- **Resistent Individuals R(t).**

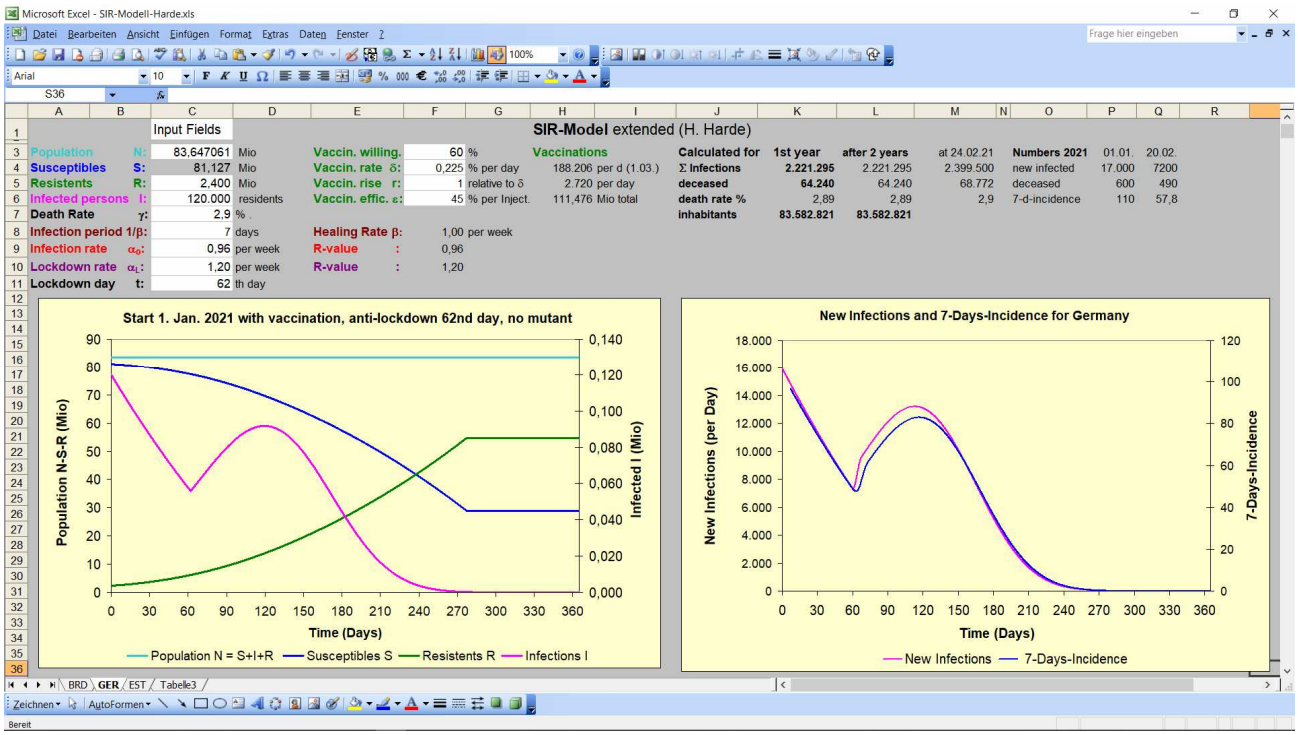
The main parameters are the:

- **infection rate α,**
- **healing rate β** resp. **infection period 1/β,**
- **death rate γ,**
- **vaccination rate δ and efficiency ε.**

$$\begin{aligned}\frac{\Delta S(t_n)}{\Delta t} &= -\alpha \cdot \frac{I(t_n)}{N(t_n)} \cdot S(t_n) - \delta \cdot \varepsilon \cdot N(t_n) \\ \frac{\Delta I(t_n)}{\Delta t} &= \alpha \cdot \frac{S(t_n)}{N(t_n)} \cdot I(t_n) - \beta \cdot I(t_n) \\ \frac{\Delta R(t_n)}{\Delta t} &= (\beta - \gamma) \cdot I(t_n) + \delta \cdot \varepsilon \cdot N(t_n)\end{aligned} \quad . \tag{13}$$

Excel-Program

The input and output fields for the calculation with Excel are shown in the upper part of the page (screen shot). Input fields are white, output fields are grey and should not be overwritten.



The graph on the left shows the three groups Susceptibles S (blue), Infected I (pink) and Resistants (green) over one year. The total population N = S + I + R , which can change due to corona deaths over the calculated period, is also displayed (aquamarine). For the susceptibles, resistants and the total number, the left ordinate is valid in units of million inhabitants, for the infected the right ordinate (also in Mio) applies, which adapts to the maximum number and enables a higher resolution.

The graph on the right shows the calculated new infections per day (moving 4-days average, pink, left ordinate) and the 7-day-incidence (blue, right ordinate), both directly as the number of people.

Data Feeding

Several starting values and parameters must be entered or changed for a calculation:

Important: If the German version is preferred for the display of decimal separators as commas and for the 1.000 separators as points, the corresponding setting must be made on the header line of the Excel file under *Extras> Options> International.*

Accordingly, values with decimal places must be separated by a comma. If the American separation is preferred, the reverse has to be done.

Start values : This includes the **population** and the number of **resistant people** and must be entered in units of millions. The input fields are to the right and are white. The initial number of **infected** is directly typed in, and from this together with the population and the resistants the number of **susceptibles** is calculated, displayed in units of millions (box C4).

Death rate γ: Apparently due to the age statistics the death rate in Germany is comparatively high with γ = 2.9%, this despite an excellent health care and clinical facilities (see column M, update from February 24). In comparison, in Estonia it is only 0.9%. Another reason might be the statistical evaluation by the Robert Koch Institute, which doesn't differentiate between deceased by and with Corona. Currently, Sweden has a death rate, which is 70x lower.

Infection period 1/β: For the calculation it is assumed that an infected person is immediately contagious over the course of the disease. In reality, however, a COVID-19 infected person is particularly contagious in the first few days after his own infection with the peak on the day immediately before the onset of symptoms. This extends over an average of 5 days. In the second week, however, he is hardly contagious, even when he often feels really sick. Therefore, it makes sense to calculate with an infection duration of 7 days, during which all infections occur. The 2nd week can largely be neglected in terms of a further infection. The respective **healing rate β** is displayed per week.

Infection rate α: The second and perhaps most important parameter, which significantly determines the course of the epidemic, can be significantly influenced by limiting contacts of people, through protective measures and hygiene concepts. A distinction is made between two infection rates, a value α_0, which is chosen at the start of the calculation (day 0), and a **lockdown rate α_L**, which is valid beginning with the specified **lockdown day**. This allows to simulate, how the course of the pandemic is influenced by restrictions, but also, how for $\alpha_L > \alpha_0$ generosity will have an acceptable or increasingly negative effect on the course. The effects of virus modifications (e.g., B 1.17) can also be simulated in this way.

Reproduction value R_0 : The ratio of the infection rate to healing rate is referred to as the base reproduction value $R_0 = \alpha/\beta$. It indicates the number of new infections that in average are caused by an infected person over the duration of the infection. For $R_0 > 1$ an epidemic spreads, for $R_0 = 1$ the number of infections remains almost constant, and for $R_0 < 1$ the infections decrease. The R values before and during a lock-down are displayed.

Vaccination Willingness: Currently, in Germany the willingness to be vaccinated is estimated to be approximately 60%. So far, however, this has no influence on the number of vaccinations carried out. Only when the total number of vaccinations (including the 2nd vaccination and efficiency) has reached the number of those ready for a vaccination, the last term in Eq. (13) becomes zero.

Vaccination rate δ: In the start-up phase, the number of vaccinations was mainly determined by the production capacities of the pharmaceutical companies and the provision of vaccines. Since January 2021in Germany it was continuously rising from 25.000 to meanwhile 188.200 per day (1. March). The latter corresponds to a vaccination rate of δ = 0,225% per day, related to the population. It is displayed as vaccinations per day in box H4 and is used as relative rate for further vaccinations.

Vaccination growth rate r: For a further vaccination rate on the same level as on March 1 the growth rate has to be chosen as r = 0. For an increasing rate as used over the last two months it hold r = 1, and for a faster increasing rate, r is larger unity. The growth rate is displayed in box H5 and the total number of injections in H6.

Vaccination efficiency ε: A 90% protective effect is only achieved with a second vaccination, for Astra-Zeneca the value is even lower. Therefore, a maximum vaccination efficiency of ε = 45% for a single injection is assumed.

Further displays: In column K (bold) the calculated total infections, which happened over the 1st year are displayed. The number of deceased persons, the respective death rate and the corrected number of inhabitants are also listed for the actual and following year.

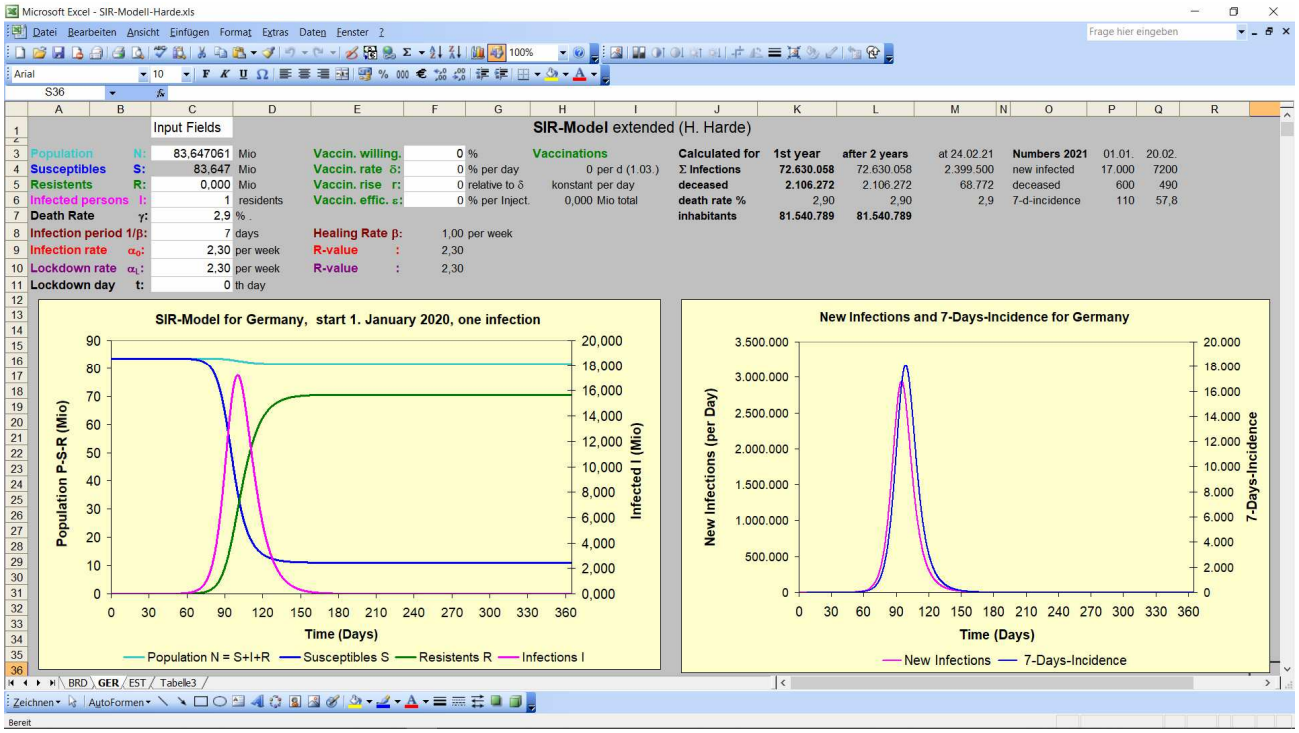
New infections: The current number of new infections per day is proportional to the modified infection rate $\alpha' = \alpha \cdot S(t)/N(t)$ and proportional to the number of people currently infected (see Eq.(7) or Eq .(8)) . These new infections per day are shown in the right diagram as a pink graph over one year, directly as number of people.

7-Days-Incidence: By many epidemiologists this value is considered to be a particularly suitable parameter for the follow-up of infections. Below a value of 50, still better below 35, it is assumed that health departments can still identify individual routes of infections and take suitable measures against increasing spread. The 7-days-incidence is the sum of new infections over the last 7 days and is based on a population of 100,000. In Estonia a 14-days incidence is used.

Examples of Simulations

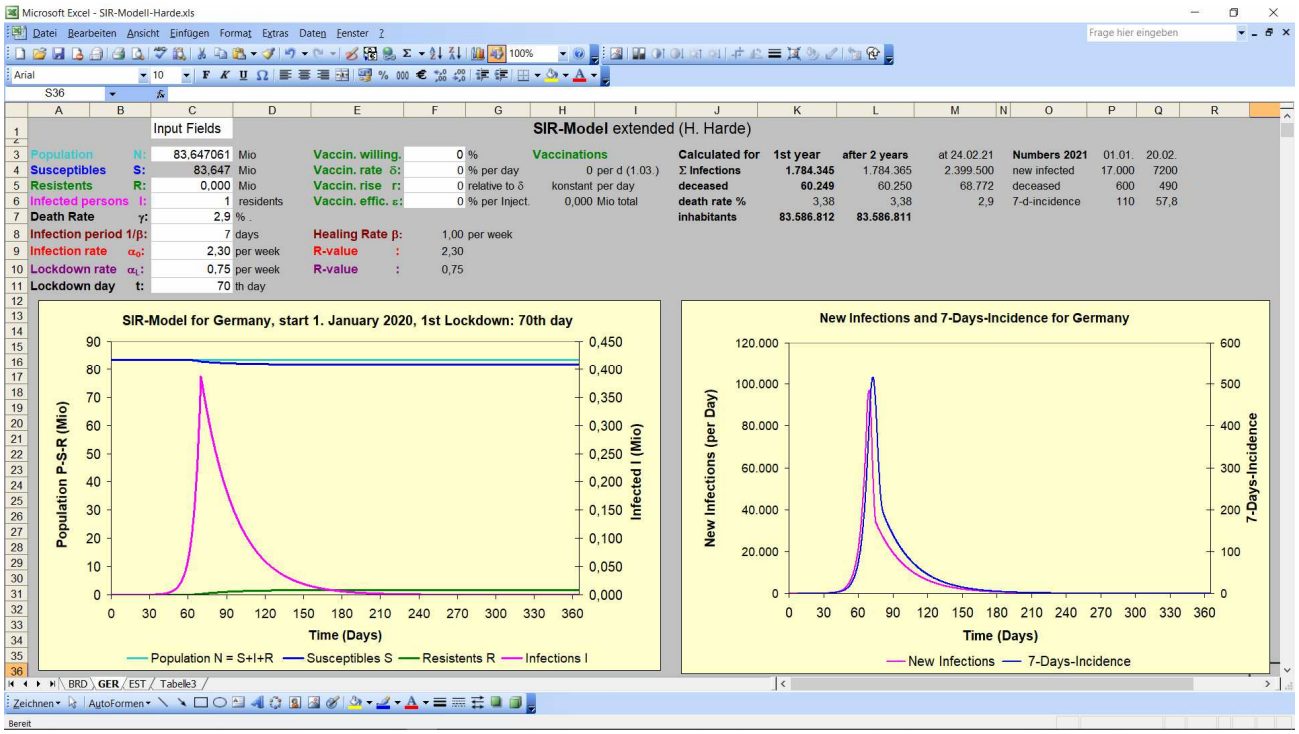
Start of the Pandemic without Protective Measures

At the start of the pandemic we assume only one infected person, an infection rate $\alpha = 2.3$ per week, a reproduction factor $R_0 = 2.3$ and no vaccinations. This describes the situation in January 2020. Unrestrained exponential increase of infected people over the first 2 1/2 months. Without lock-down and protective measures, herd immunity is achieved after 5 months (horizontal graph of S(t) = 11 million and R(t) = 70.5 million). But with a death rate of $\gamma = 2.9\%$ this leads to 2.1 million deaths. The maximum number of newly infected people per day is 3 million and the maximum 7-days-incidence is 18,000.



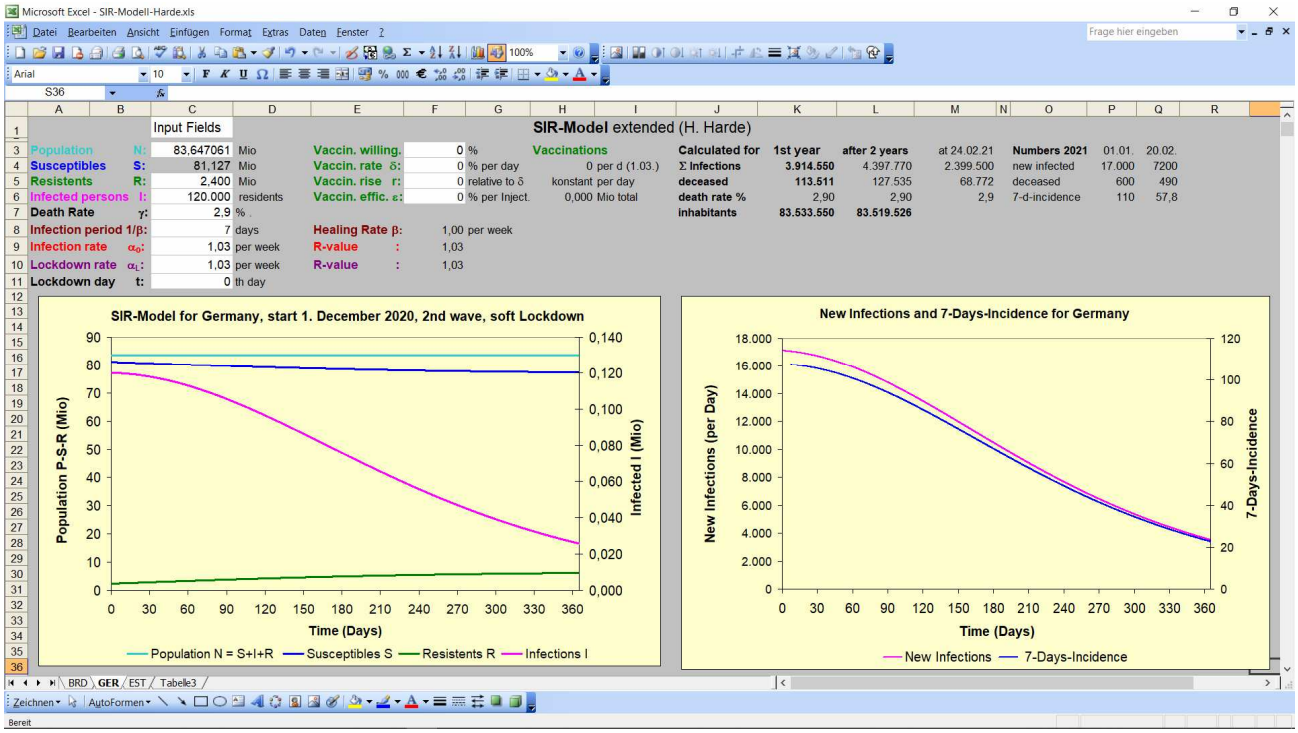
1st Tighter Lockdown after 10 Weeks (Mid March 2020)

Initial situation as before, after 10 weeks tightened lock-down: The infection rate is set to $\alpha_L = 0.75$ per week on the 70th day. The R value becomes $R_0 = 0.75$. The 7-days-incidence of 50 is reached after 47 days (about 6 weeks since onset of the lock-down). The number of deceased is 60.250. After further easing, however, the incidence rises again.



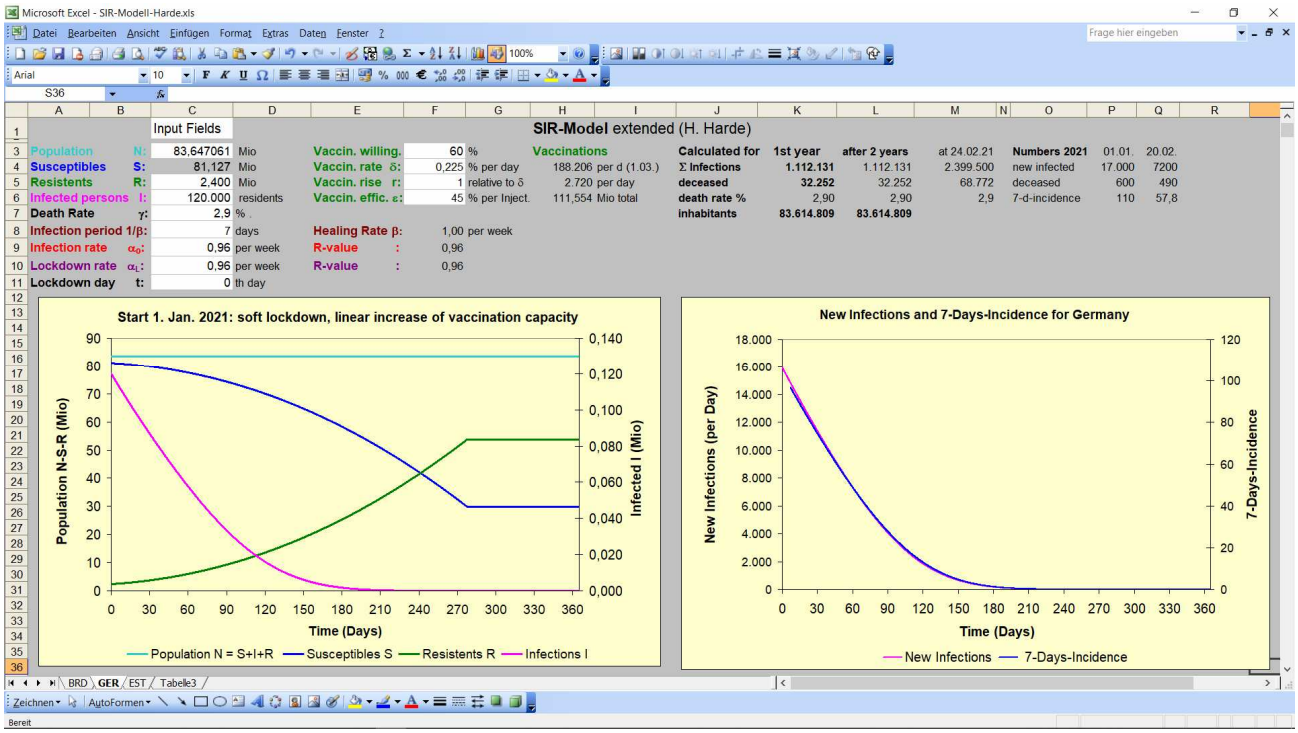
2nd Wave and Softer Lockdown since December 2020

In December we had about 120,000 infected people and more than 17,000 new infections every day. Due to those who have already been cured from previous infections, the number of resistant people is set at 2.4 million. As a result of the restricted contact measures and the closure of schools and shops, the R_0 value is about 1.03. Through December the number of infected people were almost constant, also the number resistant and susceptible persons. Under these conditions the 7-days-incidence drops from 107 to 35 in 300 days. Over 2 years there are additional 127,500 deaths. Timeline starts in December.



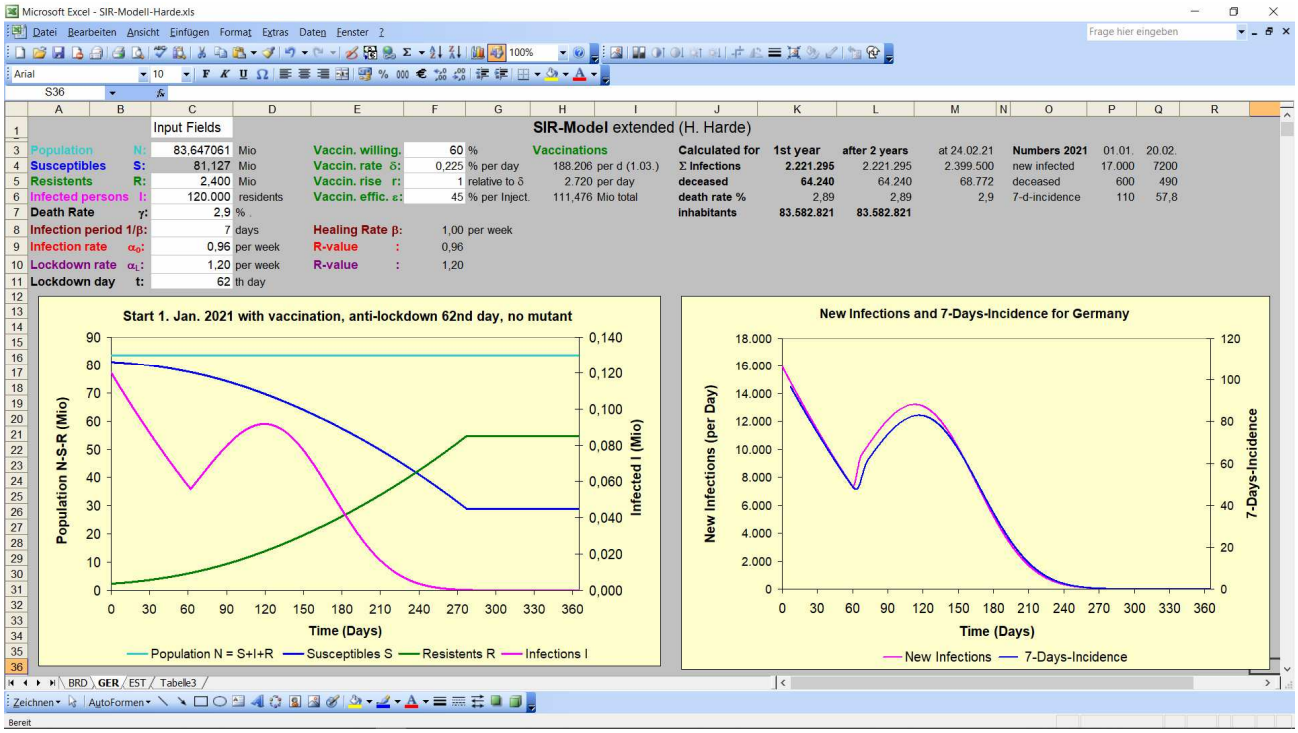
Influence of Vaccination

Vaccines have been available since the beginning of the year. With a soft lockdown (R_0 = 0.96) and a vaccination rate, which linearly increased since January and on 1st March was δ = 0.225% per day of the population (188,200 vaccinations per day), the numbers for the three groups change very significantly. These figures represent the situation since January quite well. With a further linear increase of the vaccination capacity also after the 1st March with r = 1 the 7-days-incidence decays within 2 months below 50 and is not more than 1 at mid of the year. Under these conditions the death toll runs into 32,300. Without vaccination, however, it would have increased over almost 2 years up to a total of 45,200. With a vaccination willingness of 60 % the campaign ends in September. Due to twice injections and 90% protection the efficiency is ε = 45%.



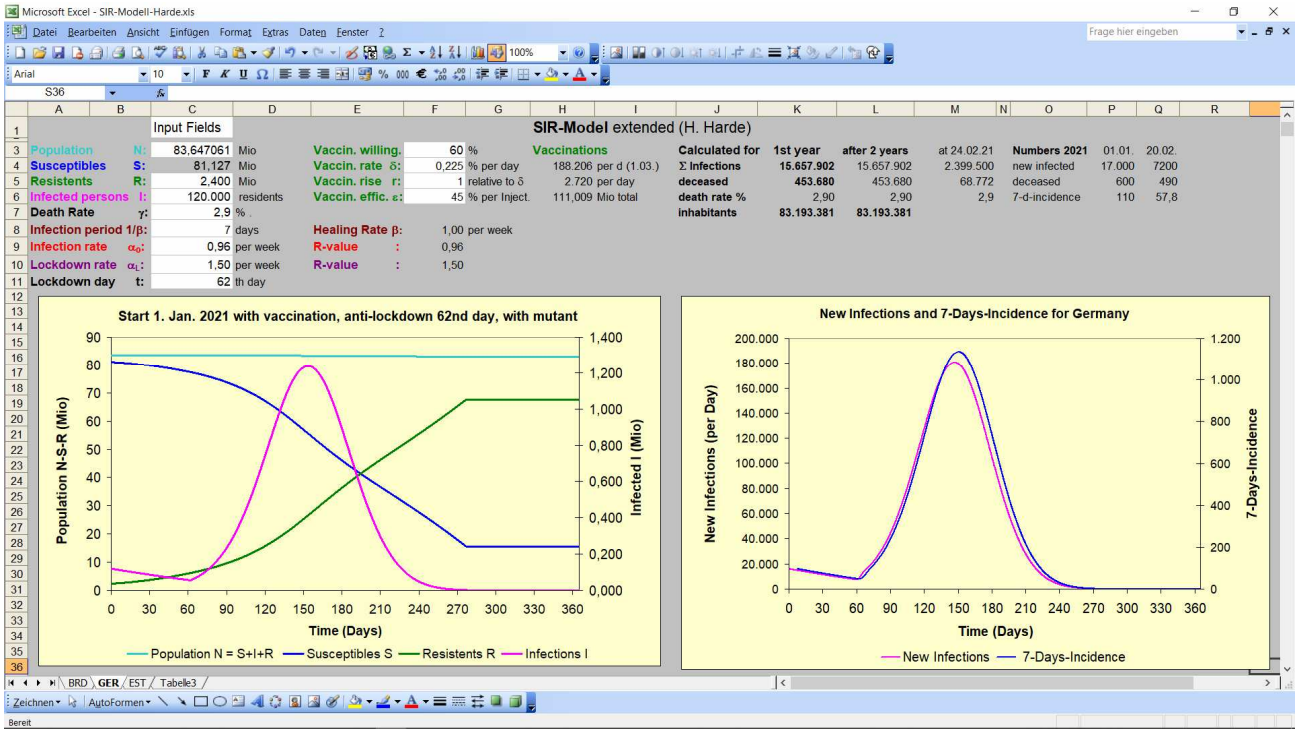
Lightening of the current restrictions, no mutant

If the same situation (R_0 = 0,96, vaccination rate δ = 0.225%/day) is assumed as in the previous case, but with the beginning of March (lock-down day = 62) a further easing of the current restrictions is considered, at least temporarily, we have to expect a new increase of infections and deceased. The time axis starts again on January 1, 2021.
With an anti-lockdown rate α_L = 1.2 per week (R_0 = 1.2), the incidence rises again up to 80 till end of April, before another drop occurs. At the same time, the number of people infected is falling significantly and the resistant group is increasing.



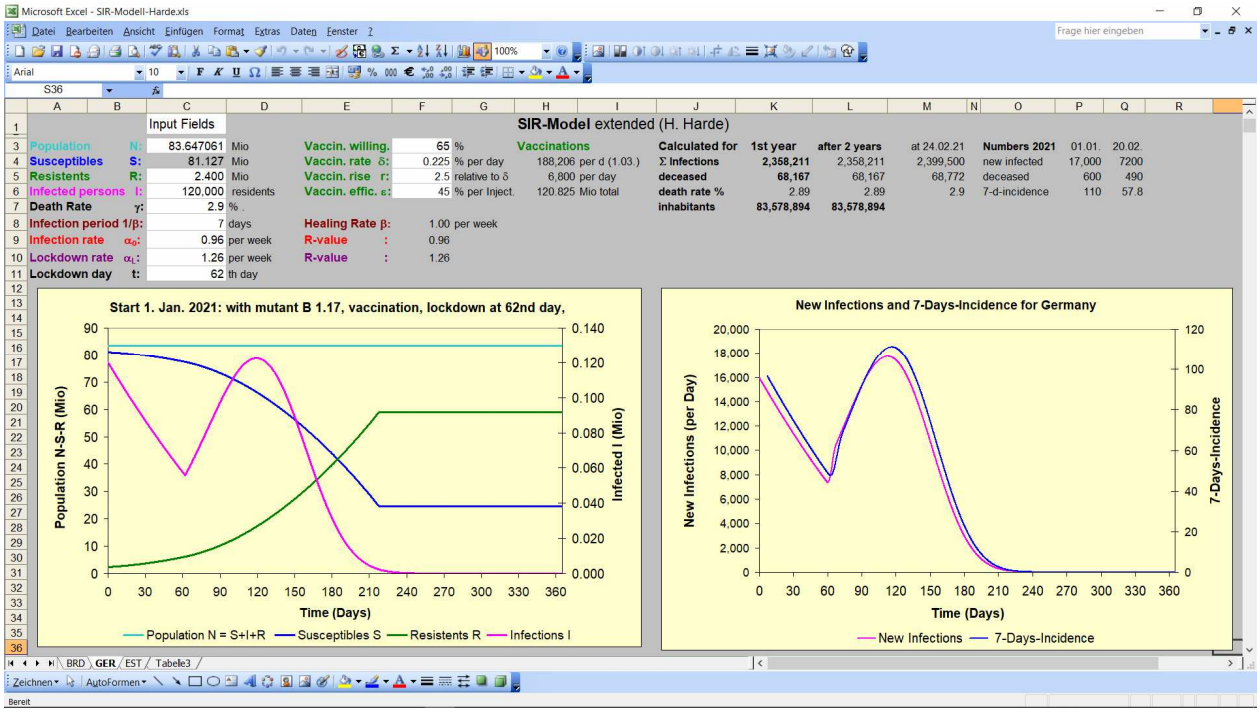
Same Easing but Additional Virus Mutant: 3rd Wave

Same conditions as before, but with virus mutant (B1.17), which has the same effect as a lightening with a higher infection rate or longer infection duration. If the infection increases by only 25%, the infection rate increases from 1.2 to 1.5 per week and R_0 = 1.5. Without any further restrictions the number of new infections increases up to 180,000 per day and the 7-days-incidence rises up to 1,100. Under these assumptions, the number of deaths at the end of the year is 454,000 plus the previous victims of about 60,000.



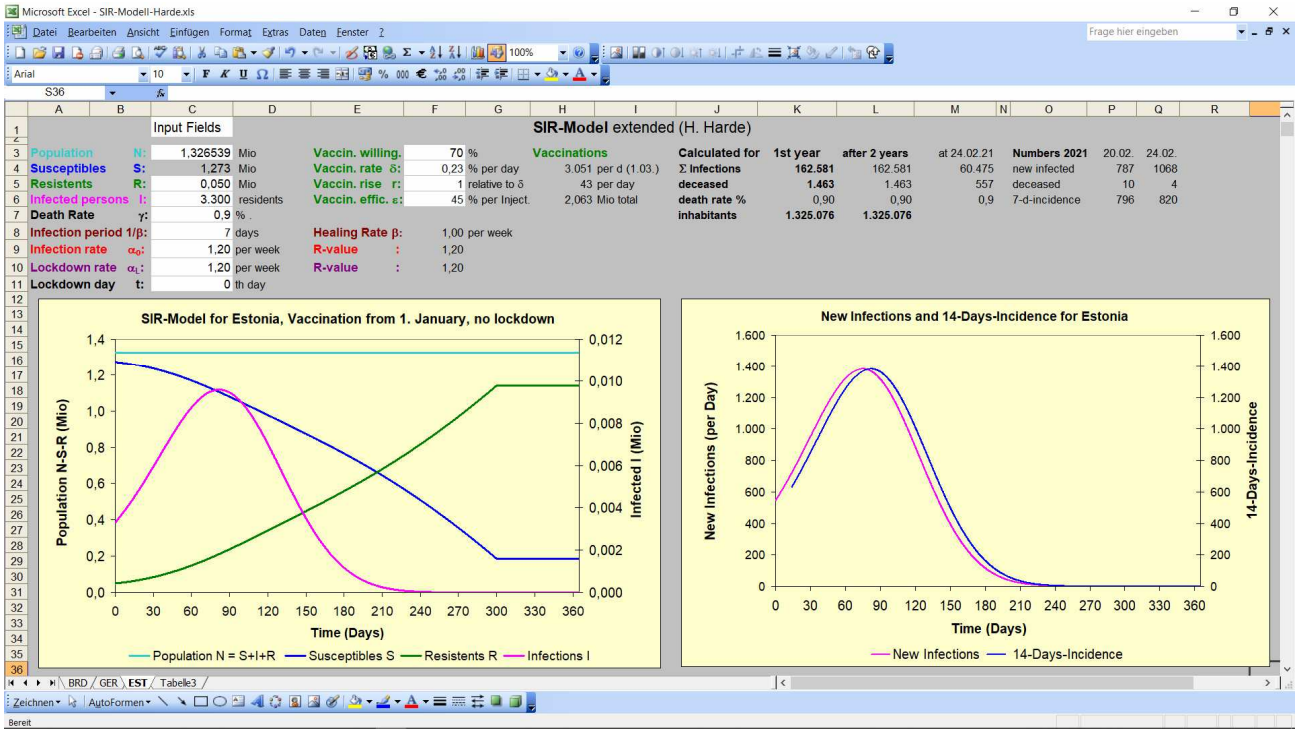
Delayed Lockdown in April 2021 and Increased Vaccination Rate

With a delayed but intensified lockdown in April and a further increased vaccination rate, which in mid June is expected to increase up to 1 Mio. per day and more (vaccination rise r = 2,5), for the mutant B 1.17 the infection rate will be about 1,26 per week (corresponds to a previous R-value of R_L = 1,0). At first this gives a further increase of new infections of 18.000 per day and an incidence of 110, before an accelerated decay in May will be observed with an incidence of about 10 at end of June. The number of deaths over the year 2021 is expected to be 68.200.



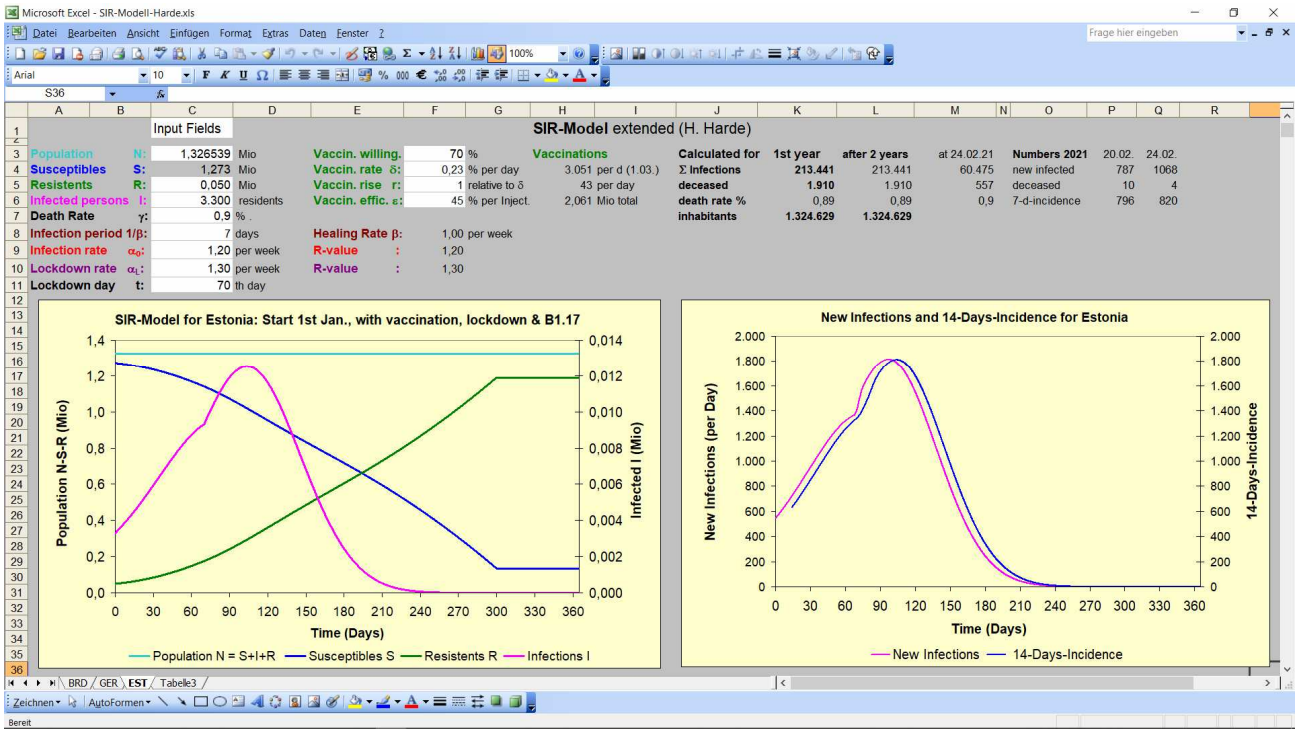
Situation in Estonia

Early in the year 2021 there were about 580 new infections per day in Estonia. On February 24 a total of 60,475 corona cases have been registered with 557 deaths and 787 new infections. The number of people recovered, is 47,000. Since then the number of new infections further inclined to 1330 and the 14-days-incidence from 796 to 1334 on March 8th.
This development reflects daily fluctuations but can well be reproduced with an infection rate of $\alpha = 1.2$ per week. Without any additional measures within one week there is only a further smaller increase of new infections up to 1,380 and within two weeks a 14-days-incidence up to 1,370. Luckily the death rate in Estonia is only around $\gamma = 0.9\%$. The vaccination rate was assumed to linearly increase up to the current value of $\delta = 0.23\%$ per day of the population. The timeline starts on the 1st January 2021. With a further linear increase of the vaccination capacities (r = 1) and without tightening measures, there will be a further 1,460 deaths by the end of the year.



With Lockdown on 11th March 2021 and Virus Mutant B 1.17

With a lockdown on the 70th day but considering the virus mutant B 1.17 the infection rate is about $\alpha_L = 1{,}3$ per week. The new infections will increase up to 1,800 per day in April and the 14-days-incidence rises also up to 1,800. The further deceased will be 1.910.



With Lockdown at 11th March, Stronger Influence of Mutant and Higher Vaccination Rate

With the lockdown at the 70th day and under the impact of mutant B 1.17 is the infection rate α_L expected to be 1,25 to 1,3 per week. The succeeding figure shows the influence of an increased vaccination rate a otherwise same conditions. The 14-days-incidence still increases up to 1.600, before it decays rapidly at mid April and approaches values of about 50 at end June. The number of deaths is about 1.550.